Appendix A

Partial Orders

A *partial order* is a relation $\leq$ acting on a set S, which satisfies

$$(x \leq y) \wedge (y \leq z) \Longrightarrow x \leq z$$

$$(x \leq y) \wedge (y \leq x) \Longleftrightarrow x = y$$

The following are examples of partial orders:

(A) The set of subsets of the natural numbers, ordered by inclusion ($\subseteq$), *e.g.*

$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4, 5\}$$

(B) The finite sequences of letters ordered lexicographically. *i.e* as a dictionary, where the first letter is most significant, e.g

$$\langle\rangle \leq \langle a\rangle \leq \langle aa\rangle \leq \langle b\rangle \leq \langle bazzz\rangle$$

An *upper bound* of a subset A of S is an element z of S such that

$$\forall\, x : A.x \leq z$$

A *least upper bound* of a subset A of S, written $\sqcup A$, is an upper bound of A, such that, for any upper bound x of A, $\sqcup A \leq x$.

In example A, above, consider the subset $S = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. The least upper bound for S is $\{1, 2, 3\}$.

A *directed set* is a nonempty subset $\Delta \subseteq S$ such that

$$\forall\, x, y : \Delta. \quad \exists\, z : \Delta. \quad (x \leq z) \wedge (y \leq z)$$

A partial order S is said to be *complete*, if it has a least element $\perp$, and every directed set $\Delta \in S$ has a least upper bound.

Example B, above, is not a complete partial order. Consider the subset

$$U = \{\langle a\rangle, \langle aa\rangle, \langle aaa\rangle, ..\}$$

U is clearly directed, yet it has no least upper bound.

Example A, however, is a complete partial order. Any subset is directed, and has a least upper bound. There is a least element – the empty set.

If S and T are two complete partial orders and $f : S \rightarrow T$, then f is said to be *monotonic* if

$$\{x, y\} \subseteq S \wedge x \leq y \Longrightarrow f(x) \leq f(y)$$

Also f is *continuous* if whenever $\Delta \subseteq S$ is directed, $\sqcup\{f(x)|x \in \Delta\}$ exists and equals $f(\sqcup\Delta)$.

Lemma 7 *Suppose S, T are complete partial orders and $f : S \rightarrow T$ is continuous. Then f is monotonic.*

Theorem 14 (Tarski) *If S is a complete partial order, and $f : S \rightarrow S$ is continuous, then f has a least fixed point (i.e., $\exists\, x : S$ such that $f(x) = x$ and, if $f(y) = y$, then $x \leq y$). This is given by*

$$\sqcup\{F^n(\perp)|n \in \mathrm{N}\}$$

Strict Partial Orders

A *strict partial order* is a relation $<$ acting on a set S, which satisfies

$$(x < y) \wedge (y < z) \Longrightarrow x < z$$

$$x < y \Longrightarrow \neg(y < x)$$

We can always construct a strict partial order from a partial order by

$$x < y \Longleftrightarrow (x \leq y) \wedge (x \neq y)$$

And we can always construct a partial order from a strict partial order by

$$x \leq y \Longleftrightarrow (x < y) \vee (x = y)$$

We say that S is *linearly ordered*, if for any $x, y \in S$ exactly one of $x < y$, $y < x$, or $x = y$ holds.